Intro to Frequency Domain Design

MEM 355 Performance Enhancement of Dynamical Systems

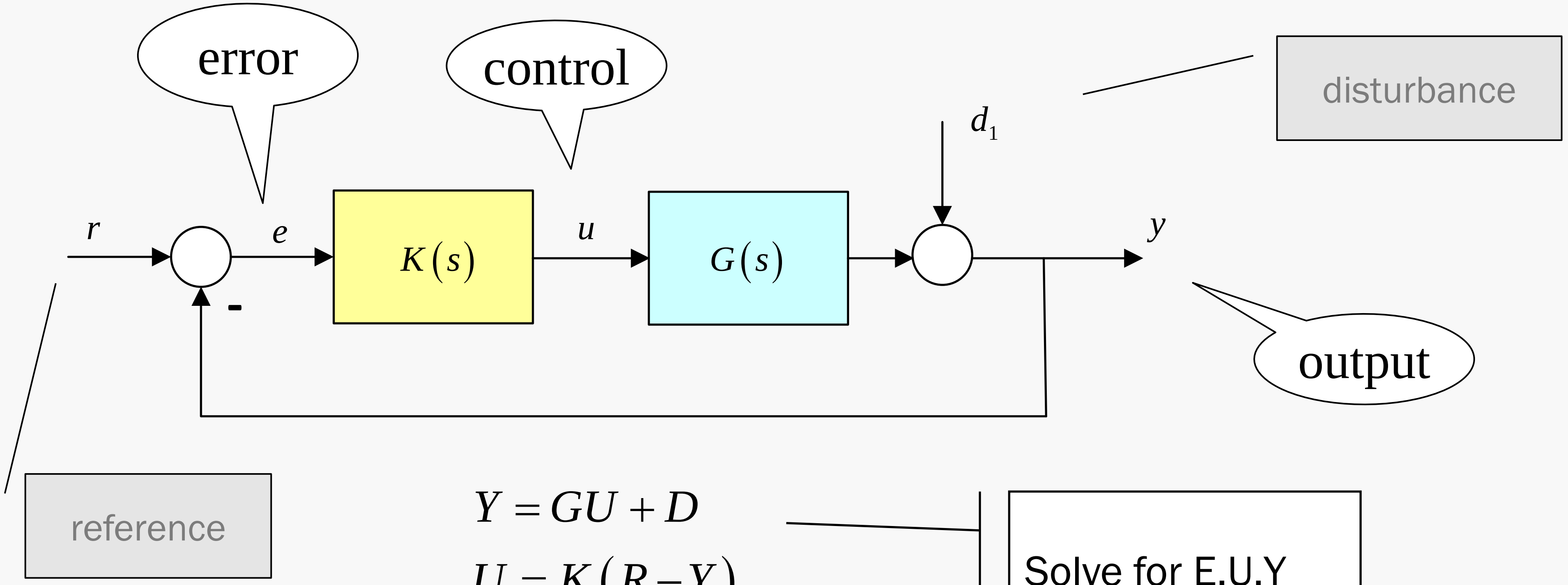
Harry G. Kwatny
Department of Mechanical Engineering & Mechanics
Drexel University

Outline

- Closed Loop Transfer Functions
  - Essential Transfer Functions
  - Sensitivity functions
- Traditional Performance Measures
  - Time domain
  - Frequency domain
- Stability & Robustness
  - Introduction – role of sensitivity functions
  - Nyquist
  - Traditional gain/phase margins

Closed Loop Transfer Functions

Closed Loop Transfer Functons
a generic structure



$$Y = GU + D$$
$$U = K(R - Y)$$
$$E = R - Y$$

Solve for E,U,Y
In terms of R,D

Transfer Functions

Error

$$E(s) = \frac{1}{1+GK} R(s) - \frac{1}{1+GK} D(s)$$

Control

$$U(s) = \frac{K}{1+GK} R(s) - \frac{K}{1+GK} D(s)$$

Output

$$Y(s) = \frac{GK}{1+GK} R(s) - \frac{GK}{1+GK} D(s)$$

Define the Loop Transfer Function:

$$L(s) = G(s)K(s)$$

$$E(s) = \frac{1}{1+L(s)} R(s) - \frac{1}{1+L(s)} D(s)$$

$$U(s) = \frac{K}{1+L(s)} R(s) - \frac{K}{1+L(s)} D(s)$$

$$Y(s) = \frac{L(s)}{1+L(s)} R(s) - \frac{L(s)}{1+L(s)} D(s)$$

Drexel UNIVERSITY

Three Key Transfer Functions

	Command	Disturbance
Error	$\frac{1}{1+L(s)}$	$\frac{-1}{1+L(s)}$
Control	$\frac{K(s)}{1+L(s)}$	$\frac{-K(s)}{1+L(s)}$
Output	$\frac{L(s)}{1+L(s)}$	$\frac{-L(s)}{1+L(s)}$

Drexel UNIVERSITY

Sensitivity Functions

$$E(s) = [I+L]^{-1} R(s) - [I+L]^{-1} D(s), \; Y(s) = [I+L]^{-1} LR(s) - [I+L]^{-1} LD(s)$$

sensitivity function: $S := [I+L]^{-1}$ command $\rightarrow$ error

complementary sensitivity function: $T := [I+L]^{-1} L$ command $\rightarrow$ output

Note the use of identity matrix, I, in these definitions. It shows the MIMO extension of the original SISO concepts.

For SISO systems Bode asked: If there is a small change in the open loop tf, L, what is the corresponding change in the closed loop command to output tf, T?

$$\frac{dT/T}{dL/L} = \frac{dT}{dL}\frac{L}{T}$$

change in command $\rightarrow$ output transfer function with respect to change in L (normalized)

$$= \left\{ -[1+L]^{-2} L + [1+L]^{-1} \right\} \frac{L}{[1+L]^{-1} L}$$

$$= -[1+L]^{-1} L + 1$$

$$= [1+L]^{-1} = S$$

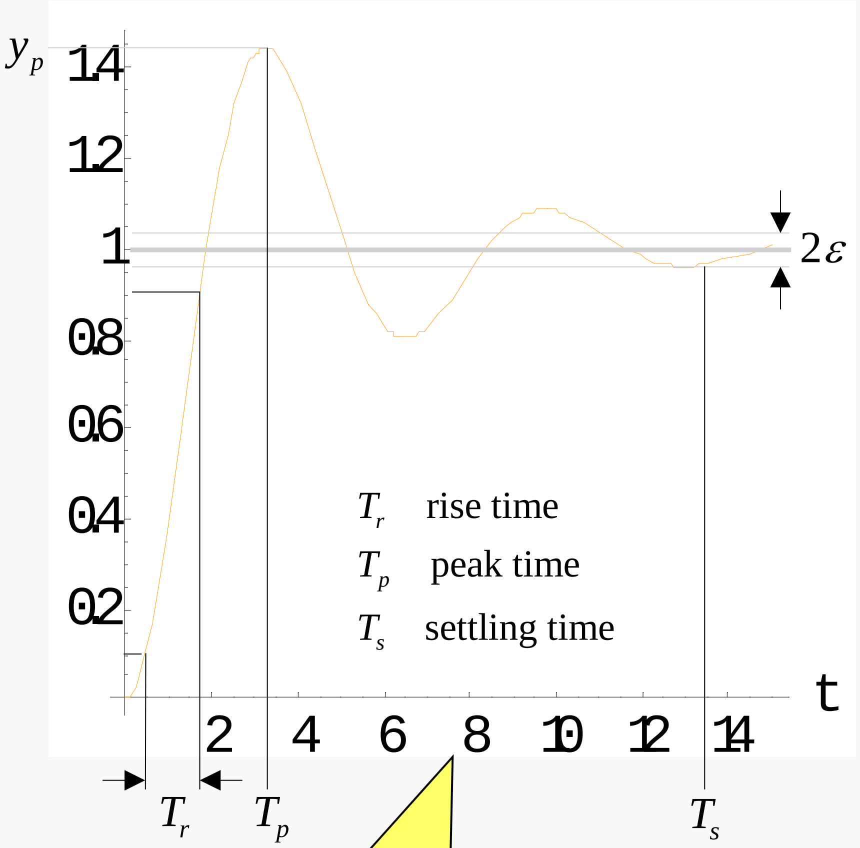
Traditional Performance Criteria

Traditional Performance ~ Time Domain

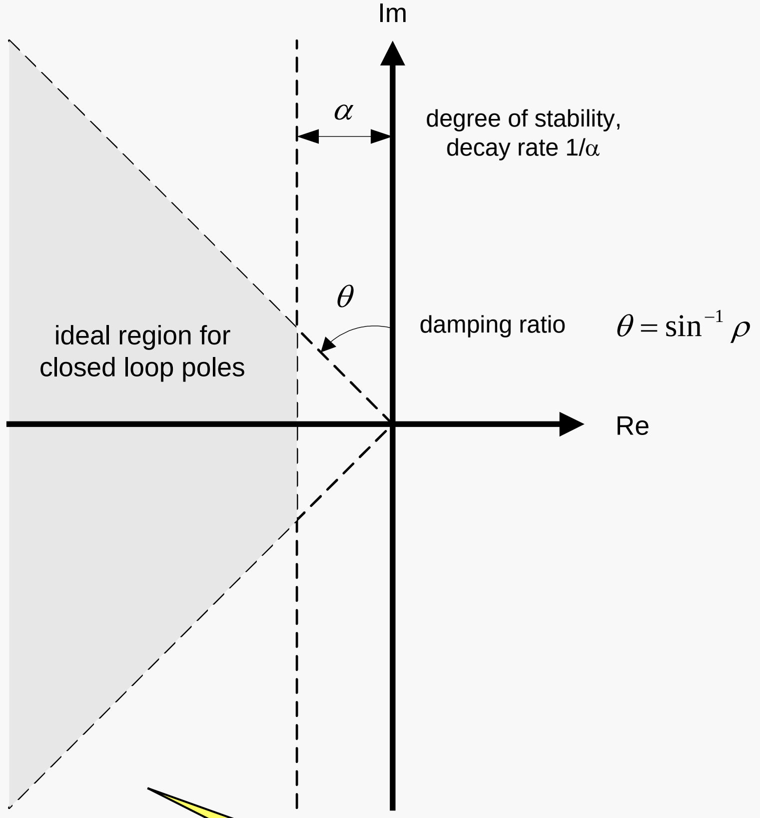
- *ultimate error*, limit of $e(t)$ as t approaches infinity
- *rise time*, T_r, usually defined as the time to get from 10% to 90% of its ultimate (i.e., final) value.
- *settling time*, T_s, the time at which the trajectory first enters an ε-tolerance of its ultimate value and remains there (ε is often taken as 2% of the ultimate value).
- *peak time*, T_p, the time at which the trajectory attains its peak value.
- *peak overshoot*, *OS*, the peak or supreme value of the trajectory ordinarily expressed as a percentage of the ultimate value of the trajectory. An overshoot of more than 30% is often considered undesirable. A system without overshoot is 'overdamped' and may be too slow (as measured by rise time and settling time).

Drexel UNIVERSITY

Traditional Performance ~ Time Domain Cont'd



Time response parameters



Ideal pole locations

Bode Plot

Given any transfer function describing a SISO system

$$Y(s) = G(s)U(s)$$

Suppose the input is a sinusoid

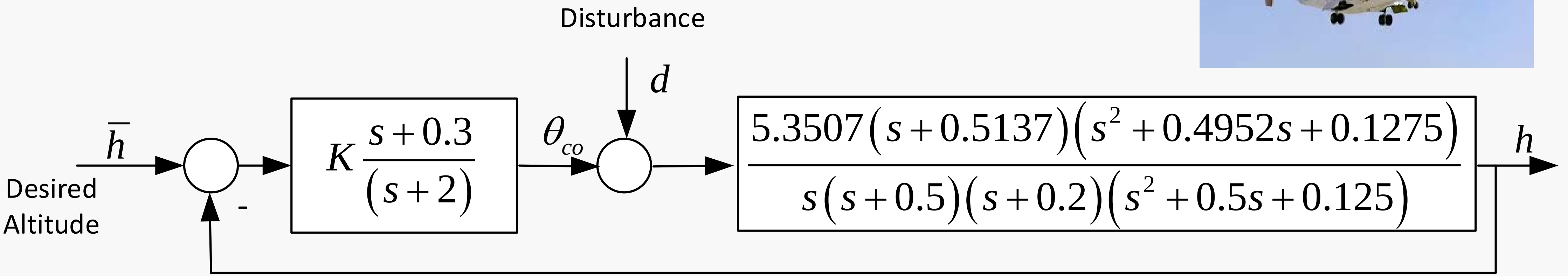
$$u(t) = A\sin(\omega t),$$

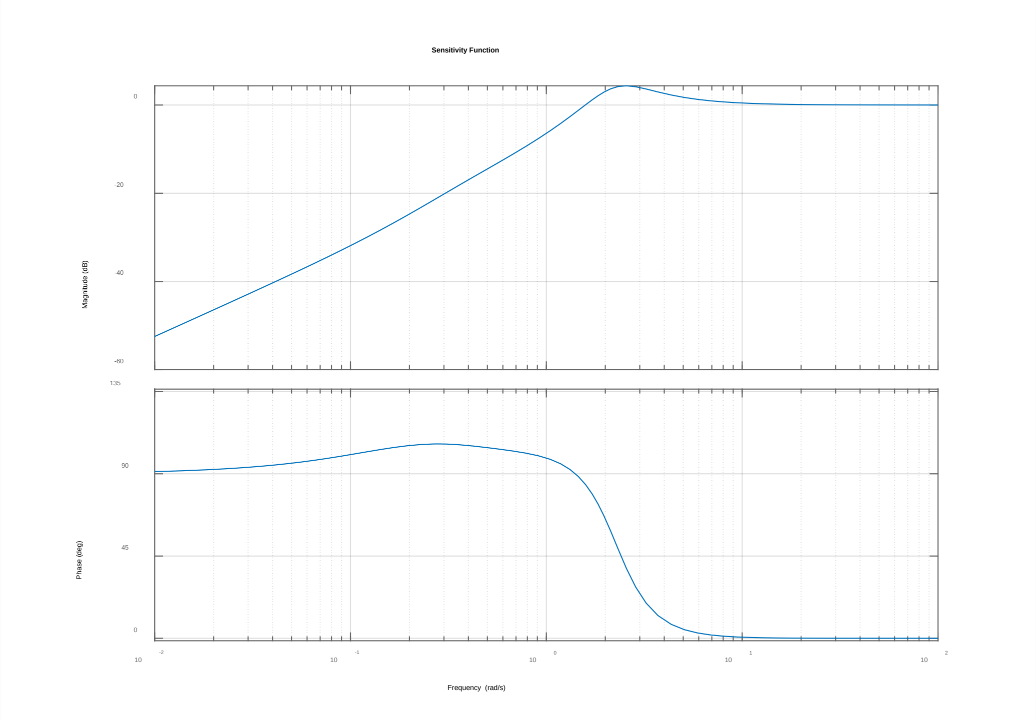
then the output is a sinusoid

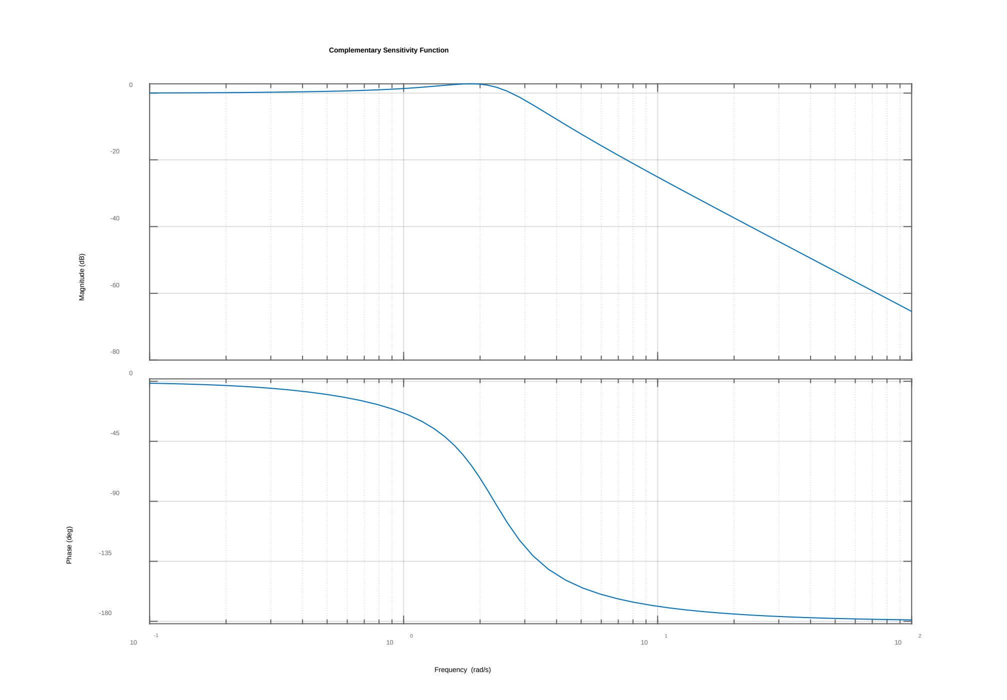
$$y(t) = B\sin(\omega t + \theta)$$

$$\boxed{B = \left|G(j\omega)\right| A, \quad \theta = \angle G(j\omega)}$$

XV-15


Disturbance
d
$\bar{h}$
Desired Altitude
-
$K\dfrac{s+0.3}{(s+2)}$
θ_{co}
$\dfrac{5.3507(s+0.5137)(s^2+0.4952s+0.1275)}{s(s+0.5)(s+0.2)(s^2+0.5s+0.125)}$
h


Sensitivity Function
Magnitude (dB)
Phase (deg)
Frequency (rad/s)


Complementary Sensitivity Function
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

Sensitivity Functions: A Fundamental Tradeoff

Note that $[1+L]^{-1}+[1+L]^{-1}L=1 \Rightarrow \boxed{S+T=1}$

Making S small improves tracking & disturbance rejection

but degrades system stability robustness and also makes it

susceptible to noise $|S|\underset{0}{\downarrow} \Rightarrow |T|\overset{1}{\uparrow}$:

Typical design specifications

$$\begin{aligned} |S(j\omega)| &<< 1 \quad \omega \in [0,\omega_1] \\ |T(j\omega)| &<< 1 \quad \omega \in [\omega_2,\infty] \end{aligned} \qquad \omega_2 > \omega_1$$

And, there are other limitations.

Sensitivity Functions, Cont'd

For unity feed back systems:

S is the error response to command transfer function

T is the output response to command transfer function

We would like S=0

We would like T=1

Suppose $L(s)$ is strictly proper $m < n,$ then

$$\lim_{\omega\to\infty} S(j\omega) = \lim_{\omega\to\infty} \frac{1}{1+L(j\omega)} = 1$$

$$\lim_{\omega\to 0} S(j\omega) = \lim_{\omega\to 0} \frac{1}{1+L(j\omega)} = \begin{cases} 0 & \text{type } L \ge 1 \\ c \ll 1 & \text{type } L = 0 \end{cases}$$

$$\lim_{\omega\to\infty} T(j\omega) = \lim_{\omega\to\infty} \frac{L(j\omega)}{1+L(j\omega)} = 0$$

$$\lim_{\omega\to 0} T(j\omega) = \lim_{\omega\to 0} \frac{L(j\omega)}{1+L(j\omega)} = \begin{cases} 1 & \text{type } L \ge 1 \\ c \approx 1 & \text{type } L = 0 \end{cases}$$

A system is of type p if the transfer function L has p free integrators in the denomintor, i.e.

$$L(s) = k\frac{s^m + a_{m-1}s^{m-1} + \cdots + a_0}{s^p\left(s^{n-p} + b_{n-p-1}s^{n-p-1} + \cdots + b_0\right)}$$

Drexel UNIVERSITY

Traditional Performance ~ Frequency Domain

Bandwidth Definitions

Sensitivity Function (first crosses $1/\sqrt{2}$=0.707~-3db from below):

$$\omega_{BS} = \max_{v}\left\{ v \ : \ \left|S(j\omega)\right| < 1/\sqrt{2} \ \ \forall \omega \in [0, v) \right\}$$

Complementary Sensitivity Function (highest frequency where T crosses $1/\sqrt{2}$ from above)

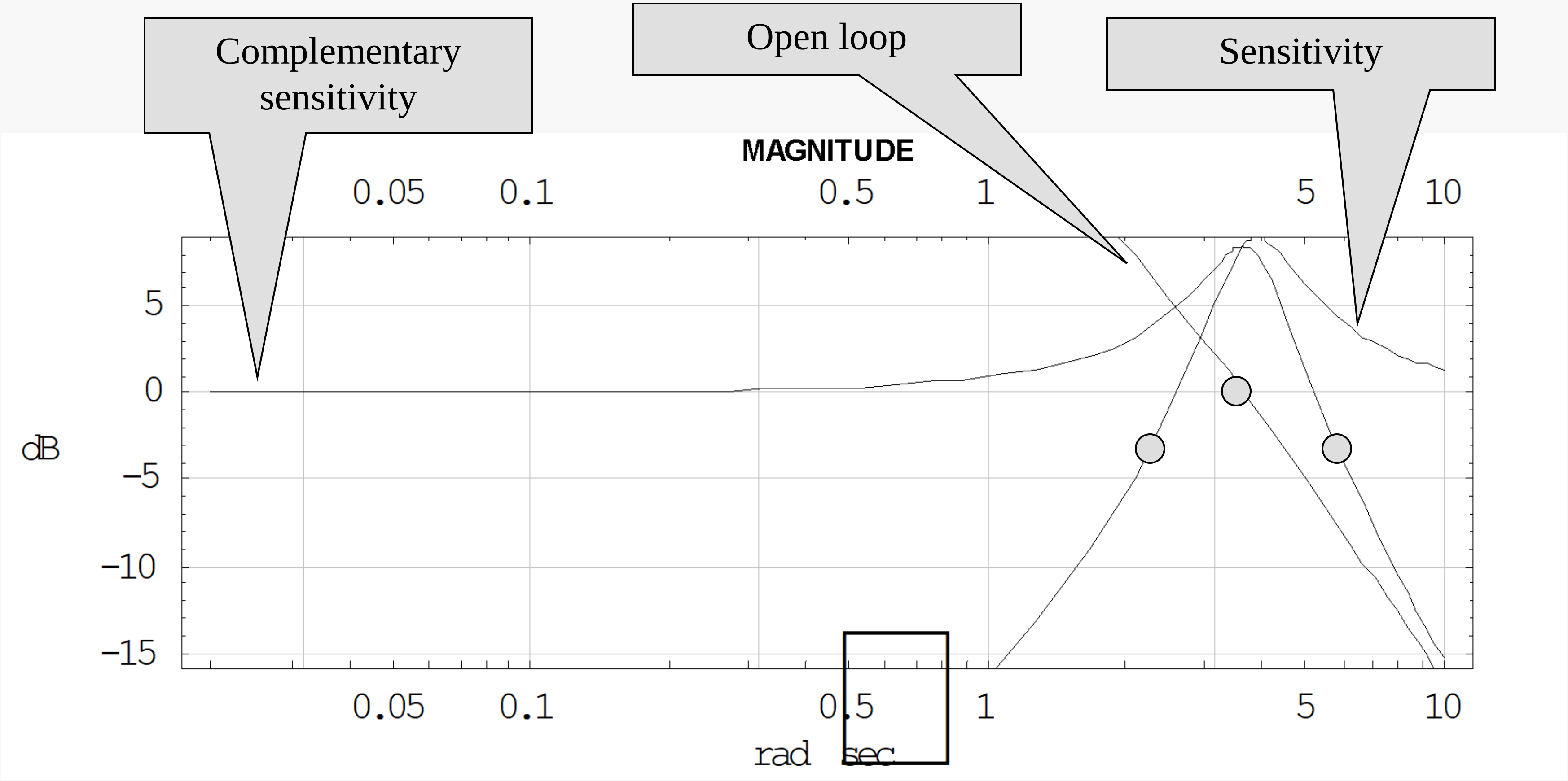
$$\omega_{BT} = \min_{v}\left\{ v \ : \ \left|T(j\omega)\right| < 1/\sqrt{2} \ \ \forall \omega \in (v, \infty) \right\}$$

Crossover frequency

$$\omega_{c} = \max_{v}\left\{ v \ : \ \left|L(j\omega)\right| \geq 1 \ \ \forall \omega \in [0, v) \right\}$$

Drexel UNIVERSITY

Example: Bandwidth



Interpretation of Bode Plot

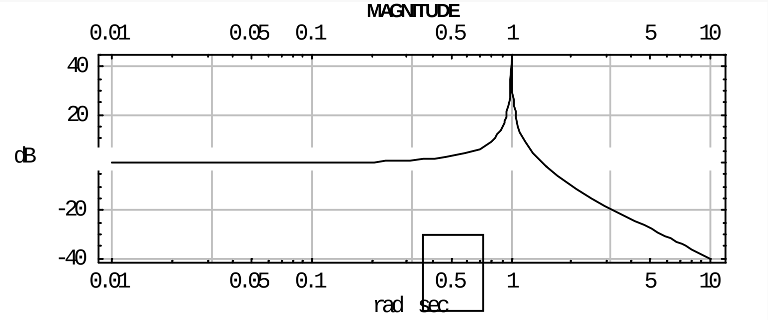
Any transfer function: $G(s)$

Output Y response to input U: $E(s) = G(s)U(s)$

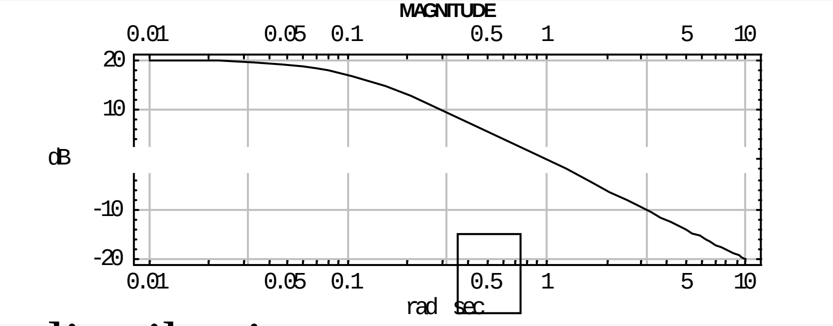
$$Y(j\omega) = G(j\omega)U(j\omega)$$

$$|Y(j\omega)| = |G(j\omega)|\,|U(j\omega)|, \quad \angle Y(j\omega) = \angle G(j\omega) + \angle U(j\omega)$$

$e^{-0.001t}\sin(\omega t)$



$e^{-0.1t}$



input frequency distribution

Summary

- Need to consider 2-3 transfer functions to fully evaluate performance
- Bandwidth is inversely related to settling time
- Sensitivity function peak is related to overshoot and inversely to damping ratio